Negative Bit Representation Outline

Negative Integers

In the first slide packet on binary representation, we saw how nonnegative integer values like 97 are represented in memory.

What if, instead of having 97, we had -97?

We need a way to represent negative integers.

Representing Negativity

For starters, we need a way to represent whether an integer is negative or positive.

We can think of this as a binary question:
a number is either negative or nonnegative.

So, we can simply pick a bit in the binary representation of the integer and decide that it's going to be the sign bit.

Which bit should we pick?

Which Bit for the Sign?

Which bit should we pick?

Well, we want to pick the bit that we're least likely to use in real life, so that it's not a big waste to use it as a sign bit.

In real life, we're much more likely to deal with very small numbers (for example, 0, 1, 13, 97) than very large numbers (for example, 4001431453).

So, we pick the leftmost bit, called the ***most significant bit***, and decide that it'll be our sign bit.

Sign-Value

Okay, now we have our sign bit.

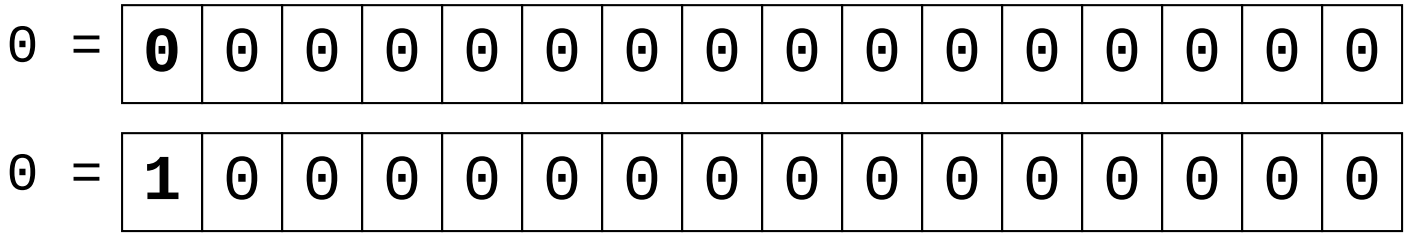
So here are three ways to represent negative integers.

Sign-Value: To get the negative version of a positive number, set the sign bit to 1, and leave all other bits unchanged:

97 =	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	1
-97 =	1	0	0	0	0	0	0	0	0	1	1	0	0	0	0	1

Disadvantages of Sign-Value

An unfortunate feature of Sign-Value representation is that there are two ways of representing the value zero: all bits set to zero, and all bits except the sign bit set to zero.



This makes the math a bit confusing.

More importantly, when performing arithmetic operations, we need to treat negative operands as special cases.

One's Complement

One's Complement: To get the negative version of a positive number, invert (***complement***) all bits; that is, all 1's become 0's and vice versa.

97 =	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	1
-97 =	1	1	1	1	1	1	1	1	1	0	0	1	1	1	1	0

Disadvantages of One's Complement

An unfortunate feature of One's Complement representation is that there are two ways of representing the value zero: all bits set to zero, and all bits set to one.

0 =	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0 =	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

This makes the math a bit confusing.

More importantly, when performing arithmetic operations, we need to treat negative operands as special cases.

Two's Complement

Two's Complement: To get the negative version of a positive number, invert all bits **and then add 1**; if the addition causes a ***carry*** bit past the most significant bit, discard the high carry:

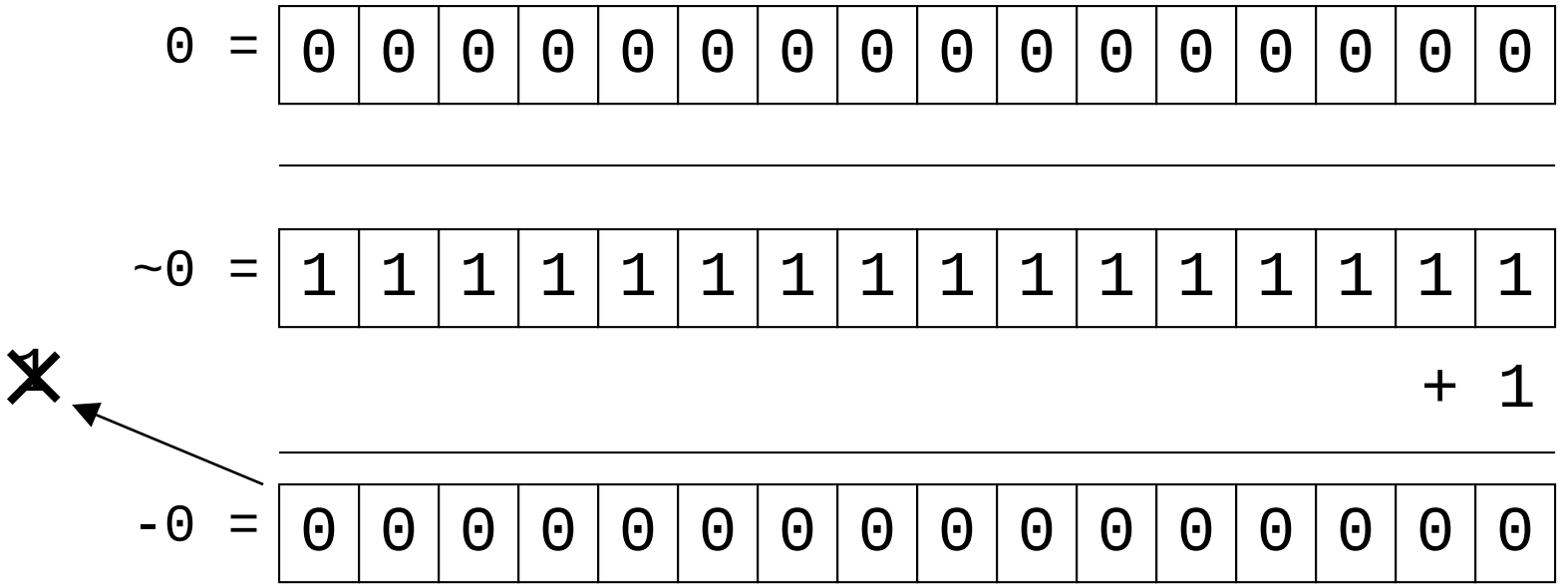
```
 97 = | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 1 |

~97 = | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 1 | 1 | 1 | 1 | 0 |
                                                                    + 1

-97 = | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 1 | 1 | 1 | 1 | 1 |
```

Advantages of Two's Complement #1

In Two's Complement representation, the value zero is uniquely represented by having all bits set to zero:



Advantages of Two's Complement #2

When you perform an arithmetic operation (for example, addition, subtraction, multiplication, division) on two signed integers in Two's Complement representation, you can use **exactly the same method** as if you had two unsigned integers (that is, nonnegative integers with no sign bit) ...

… **EXCEPT**, you throw away the high carry (or the high **borrow** for subtraction).

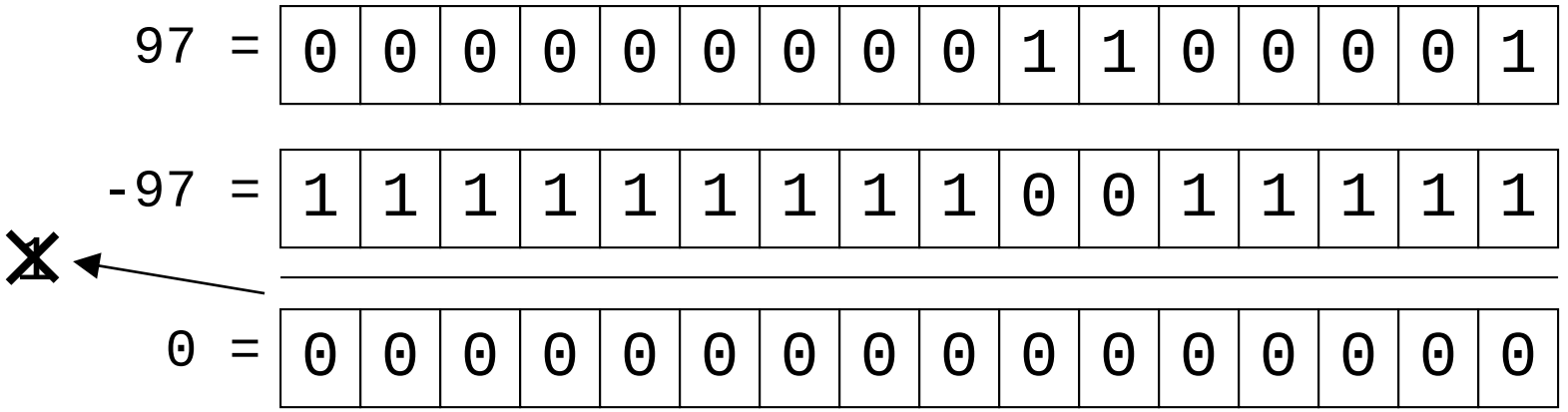
Advantages of Two's Complement #3

This property of Two's Complement representation is so incredibly handy that virtually every general-purpose computer available today uses Two's Complement.

Why? Because, with Two's Complement, we don't need special algorithms (and therefore extra circuitry) for arithmetic operations that involve negative values.

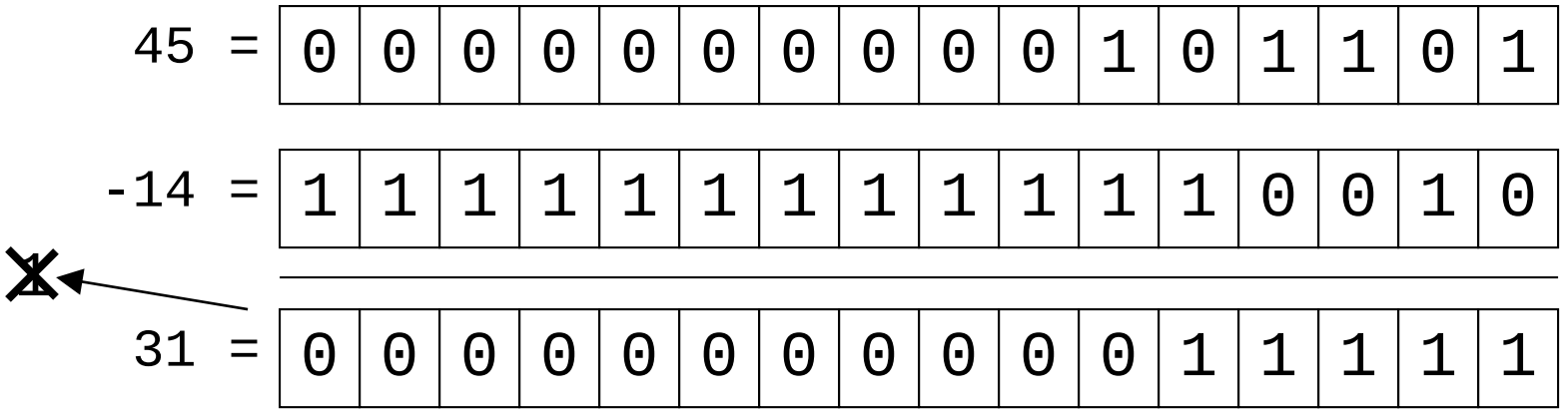
Advantages of Two's Complement #4

Using Two's Complement to do arithmetic:



Advantages of Two's Complement #4

Using Two's Complement to do arithmetic:



Range of Two's Complement Values #1

When we represent negative integers in Two's Complement notation, the range of numbers that can be represented in *b* bits is:

$$-(2^{b-1}) \ldots (+2^{b-1} - 1)$$

For example, the range of numbers that can be represented in 8 bits is:

$$-(2^{7}) \ldots (+2^{7} - 1) = -128 \ldots 127$$

Likewise, the range of numbers that can be represented in 16 bits is:

$$-(2^{15}) \ldots (+2^{15} - 1) = -32{,}768 \ldots 32{,}767$$

How do we know this?

Range of Two's Complement Values #2

When we represent negative integers in Two's Complement notation, the range of numbers that can be represented in *b* bits is:

$$-(2^{b-1}) \ldots (+2^{b-1} - 1)$$

How do we know this?

Here's the biggest number that can be represented in 16 bits:

0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

If we add one to it, we get:

1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

But this number is negative (that is, it has a 1 in the sign bit).

Range of Two's Complement Values #3

Here's the biggest number that can be represented in 16 bits:

0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

If we add one to it, we get:

1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

But this number is negative.

If this number were "unsigned", then it'd be 2^{15}, which is 2^{16-1}, which is 2^{b-1}, where *b* (the number of bits) is 16.

Therefore, the largest number that can be represented in *b* bits in Two's Complement must be $2^{b-1}-1$.

Range of Two's Complement Values #4

So what's the smallest negative integer (that is, the negative integer with the greatest absolute value)?

Well, can we represent the negative of $2^{b-1} - 1$?

$+(2^{b-1}-1) =$	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$\sim(2^{b-1}-1) =$	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
														+	1
$-(2^{b-1}-1) =$	1	0	0	0	0	0	0	0	0	0	0	0	0	0	1

Range of Two's Complement Values #5

We can represent $-(2^{b-1} - 1)$.

So, can we represent the negative of $-(2^{b-1})$?

$-(2^{b-1} - 1) =$

1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
														-	1

$-(2^{b-1}) =$

1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Since the sign didn't change, it worked!

But, if we tried subtracting again, we'd borrow and the sign would change, indicating failure.

So, $-(2^{b-1})$ is the lowest number that can be represented (that is, the negative number with the highest absolute value).

Overflow #1

When we're working with a value that's near the upper limit of what can be represented in Two's Complement for the given number of bits, we sometimes perform an operation that should result in a positive value but instead produces a negative value.

Such an event is called ***overflow***.

Overflow #2

Consider the following addition in 8-bit Two's Complement representation:

+127 =	0	1	1	1	1	1	1	1
+ 1								+ 1
+128? =	1	0	0	0	0	0	0	0

Notice that the result should be +128,
but because the leftmost bit is 1, it's actually -128.

This is ***overflow***: an arithmetic operation that should have a positive result goes over the top to become negative.

Underflow #1

When we're working with a value that's near the lower limit of what can be represented in Two's Complement for the given number of bits, we sometimes perform an operation that should result in a negative value but instead produces a positive value.

Such an event is called ***underflow***.

Underflow #2

Consider the following addition in 8-bit Two's Complement representation:

-128 =	1	0	0	0	0	0	0	0
- 1								- 1
-129? =	0	1	1	1	1	1	1	1

Notice that the result should be -129, but because the leftmost bit is 1, it's actually +127.

This is ***underflow***: an arithmetic operation that should have a negative result goes under the bottom to become positive.

Overflow Example

"17. Inserted debug statements into anomdtb.f90, discovered that a sum-of-squared variable is becoming very, very negative! Key output from the debug statements:

```
OpEn=   16.00, OpTotSq=    4142182.00, OpTot= 7126.00
DataA val =      93, OpTotSq=       8649.00
DataA val =     172, OpTotSq=      38233.00
DataA val =     950, OpTotSq=     940733.00
DataA val =     797, OpTotSq=    1575942.00
DataA val =     293, OpTotSq=    1661791.00
DataA val =      83, OpTotSq=    1668680.00
DataA val =     860, OpTotSq=    2408280.00
DataA val =     222, OpTotSq=    2457564.00
DataA val =     452, OpTotSq=    2661868.00
DataA val =     561, OpTotSq=    2976589.00
DataA val =   49920, OpTotSq=-1799984256.00
DataA val =     547, OpTotSq=-1799684992.00
DataA val =     672, OpTotSq=-1799233408.00
DataA val =     710, OpTotSq=-1798729344.00
DataA val =     211, OpTotSq=-1798684800.00
DataA val =     403, OpTotSq=-1798522368.00
OpEn=   16.00, OpTotSq=-1798522368.00,
OpTot=56946.00
forrtl: error (75): floating point exception
IOT trap (core dumped)"
```

"..so the data value is unbfeasibly large, but why does the sum-of-squares parameter OpTotSq go negative?!!

Probable answer: the high value is pushing beyond the single-precision default for Fortran reals?"

http://www.blc.arizona.edu/courses/schaffer/182h/Climate/Harry_Read_Me%20%28Page%20Numbers%29.pdf

Overflow Example (cont'd)

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"..so the data value is unbfeasibly large, but why does the sum-of-squares parameter OpTotSq go negative?!!

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Likely actual answer:

The code used an integer to do the arithmetic, then multiplied by a floating point number before outputting -- but it got an overflow.

http://www.blc.arizona.edu/courses/schaffer/182h/Climate/Harry_Read_Me%20%28Page%20Numbers%29.pdf